



TE TAI ŌHANGA
THE TREASURY

New Zealand Superannuation Fund (NZSF) Model Guide

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1 The purpose of the New Zealand Superannuation Fund (NZSF)

The majority of New Zealanders aged 65 years and above receive the public pension, New Zealand Superannuation (NZS).¹ This is also referred to as NZ Super, or even just Super. NZS recipients are often referred to as superannuitants. NZS payments, which have tax deducted from them, are made on a fortnightly basis and are met from general tax revenue. In other words, the present situation is that taxpayers cover the cost of NZS each year.

However, New Zealand's population structure has been ageing in recent decades, as is the case for many other countries. Furthermore, the demographic projections of Stats NZ expect this to continue until at least the end of this century, and quite likely beyond that. This means that the number of superannuitants is growing relative to the size of the taxpayer base funding NZS. This demographic shift is at the heart of the tax smoothing principle that is the fundamental role of the NZSF.

The main objectives for setting up the NZSF were described in a working document published by the Treasury in June 2000, titled [Pre-funding New Zealand Superannuation](#). These were to:

- increase national saving
- increase future growth
- redistribute income from present to future taxpayers
- enhance the Crown's capacity to withstand unexpected future events
- bolster the credibility and stability of New Zealand Superannuation policy, and
- better manage the tension between short- and long-term demands on government finances.

Between the fiscal years 2001/02² and 2009/10 and then again from 2017/18 onwards, the government has contributed funds each year, called capital contributions, to the NZSF so that it can invest and grow its assets. Even in the years 2010/11 to 2016/17, when no capital contributions were made, the NZSF was still, of course, investing and growing.

The purpose of this building up of the NZSF is that, starting from some future date, the government of that time will be able to receive funds each year from the NZSF, called capital withdrawals, rather than provide it with funds. These contributions will help taxpayers of that time cover the costs of NZS, which will have risen not just in nominal dollar terms (which would happen normally with just the influence of inflation) but, more significantly, as a percentage of gross domestic product (GDP). This lift relative to GDP, which is referred to in economic terms as an increase in the real cost of NZS, is due to the ageing population structure causing the recipient base of NZS to grow at a faster rate than the taxpayer base.

¹ There are qualification requirements of having resided in New Zealand for at least ten years, of which at least five have to have been since attaining the age of 50 years. There are also some people who, for various reasons, choose not to receive NZS despite being legally eligible.

² The New Zealand government's financial or fiscal year runs from 1 July to the following 30 June.

The NZSF acts as a tax smoothing vehicle between current and future generations of taxpayers. What this means is that current (and near future) taxpayers actually contribute more than the cost of today's NZS, as they also pay for the capital contributions to the NZSF, so that future generations of taxpayers can contribute less than the much higher costs of NZS that they will face, because they are aided by the capital withdrawals from the NZSF.

The [New Zealand Superannuation and Retirement Income Act \(2001\)](#) (the Act) established the NZSF. It is also sometimes referred to as the NZ Super Fund or the Cullen Fund, after Sir Michael Cullen, the Minister of Finance who established the NZSF.

The management and administration of the NZSF is done by an autonomous Crown entity, the [Guardians of New Zealand Superannuation](#), which was also established by the Act.

2 The role of the New Zealand Treasury

The Treasury is responsible, as defined in [Section 42](#) of the Act, for calculating the amount of the annual capital contribution in each fiscal year, or capital withdrawal once these begin in future years. All assumptions and judgements used in this calculation must be stated in the *Economic and Fiscal Update* (EFU) in which the contribution is published.

[Section 43](#) of the Act specifies how the calculation is to be done. The primary purpose of the NZSF model is to perform this calculation, and the majority of the model's assumptions, inputs and computational logic are concerned with working out the contribution rate in each year. This is the variable denoted as "a" in the formula in Section 43.

As required by the legislation, the contribution rate, as a percentage of GDP in any year, is the sum of the capital contribution (or, in later years, the capital withdrawal), and the aggregate amount of net (after-tax) NZS expenditure, divided by nominal GDP. All three components need to be in the same units, which is normally billions of New Zealand dollars.

The Treasury maintains, develops and runs the NZSF model, which is an Excel spreadsheet. Most of the inputs to the model are produced by the Treasury, such as the long-range GDP and aggregate net NZS expenditure tracks, using some of its other models. The NZSF model also makes use of forecasts from both the NZSF itself and from the Ministry of Social Development (MSD) and utilises the demographic projections of Stats NZ.

3 Clarification of terms used

Members of the public, media and even government officials sometimes refer to the NZSF when they mean NZS, and vice-versa, which is not surprising given the similarity of the abbreviations and the fact that New Zealand Superannuation appears in both names.

However, while the whole purpose of the NZSF is to help future taxpayers cover the increasing cost of NZS, it is important to understand that the NZSF itself:

- does not pay out NZS to individual recipients, as that role is fulfilled by the operational arm of MSD, which is Work and Income New Zealand (WINZ)
- does not hold any individual's retirement savings accounts, and
- is in no way involved in the day to day administration of public pension payments, nor in the development of policy concerning NZS.

It is also worth clarifying that MSD has no involvement in the administration, investment strategy, decision making or indeed any aspect of the running of the NZSF.

In this guide, unless otherwise stated, any reference to GDP should be taken to mean nominal gross domestic product. Likewise, any reference to Net NZS should be taken to mean the total or aggregate amount of expenditure on the public pension, New Zealand Superannuation, after deducting the aggregate amount of tax paid on this pension.

Rather than constantly repeat the term worksheet, this guide will often just refer to the worksheets by name (in italics) eg, "in *Input*" rather than "in the *Input* worksheet".

The NZSF model begins its calculations following the last completed fiscal year for which outturns of the parameters of the NZSF are available ie, results are known for the capital contribution, revenue, tax expense, other expenses, gains or losses, other movements in reserves and closing balance of the NZSF. This year is referred to in this guide as **Year 0**.

4 Overview of the NZSF model

The NZSF model is an Excel spreadsheet, consisting of several worksheets. Of these, only *Input* and *Model* are essential to operating the NZSF model and producing its outputs, which include the calculations of the capital contributions to and withdrawals from the NZSF.

Table 1 – Description of worksheets in the NZSF model

Worksheet	Description of the worksheet
<i>Information about the model</i>	This contains information about this latest version of the NZSF model. It explains the purpose of each worksheet in the model. A list of the assumptions that can be changed in <i>Input</i> is provided. These can be used to produce different scenarios for future outcomes of the NZSF. The values of the key standard assumptions in <i>Input</i> are also stated.
<i>NZS data</i>	Data is provided about the expenditure on and recipient numbers of the public pension, NZS. The data, which is in fiscal years, includes: <ul style="list-style-type: none"> • Historical, known outturns back to 1996/97 • the latest forecasts for the next five years from MSD, and • the post-forecast projections produced by the Treasury. Aggregate gross and net (of tax) figures are provided, both in nominal dollar terms and as a percentage of GDP. Gross NZS expenses are reported because these are how the expenses are recorded in the government accounts and it makes them comparable to how other types of public spending, like health expenditure, is reported. However, superannuitants pay tax on their NZS payments. This means that the true cost to the government, and hence to taxpayers, of funding NZS is the aggregate net expenditure, not the gross. The difference is part of tax revenue. For this reason the NZSF model uses aggregate net NZS expenditure, not gross, in its legislated formula for calculating capital contributions and withdrawals.
<i>NZSF history</i>	Historical data on the earnings, tax and non-tax expenses, capital contributions from the government and closing balances of the NZSF are provided in this worksheet. The data, in fiscal years covering from the inception of the NZSF in 2001/02 up until the latest completed fiscal year, is recorded in units of billions of New Zealand dollars.
<i>Input</i>	All input data to the NZSF model is entered via this worksheet. If any of the standard input assumptions, including the long-run input tracks of GDP and Net NZS, are altered in order to run alternative outcomes for the NZSF, then they can be restored to the latest forecast round values by picking them up from <i>Standard inputs</i> .

Worksheet	Description of the worksheet
Model	<p>All of the major fiscal variables in the NZSF model, such as capital contributions (and, in later years, withdrawals), earnings, tax, closing balance etc. are calculated and displayed here in fiscal year tracks. Data is given in both nominal dollars and as a percentage of GDP.</p> <p>The opening year is the latest completed fiscal year and the only variable needed from this year is the NZSF closing balance. Five years of forecasts follow this.</p> <p>Other than the capital contributions in at least some of the forecast years and the calculated closing balance, all of the key variables in forecast years are normally picked up from <i>Input</i>. They generally are sourced from analysts at the NZSF.</p> <p>It is in the post-forecast projected years that the modelling logic and formulae in this worksheet are applied to calculate all NZSF variables out over an extended time frame. The calculations are performed using the data and assumptions given in <i>Input</i>. It is the formulae in <i>Model</i> that are the main focus of this guide.</p>
Standard inputs	<p>This worksheet contains all the standard inputs used in <i>Input</i> for the latest official version of the model. This includes the long-run input tracks for GDP and Net NZS. The data is based on the Treasury forecasts and projections produced as part of the most recent EFU.</p>

There are also four graphs in separate worksheets in the NZSF model. Except for the *NZS to GDP* graph, the graphs derive their data from *Model*.

Table 2 – Description of graphs in the NZSF model

Graph	Description of the graph
Contribution rate	<p>This is a graph of the relative paths, as percentages of GDP, of the contribution rate, calculated from the legislated formula, and Net NZS. These paths are shown over an extended period of at least 60 years.</p> <p>When the contribution rate line is above the Net NZS line, contributions are made to the NZSF. When the contribution rate line is below that of the Net NZS, the NZSF makes payments to the government to assist future taxpayers with the rising cost of NZS.</p>
Capital contribution	<p>The long run path of capital contributions and, in later years, capital withdrawals, as a percentage of GDP, is shown in this graph.</p>
NZSF balance	<p>This is a graph of the extended path of the size of NZSF assets, shown by the closing balance in each fiscal year as a percentage of GDP.</p>
NZS to GDP	<p>Aggregate gross NZS expenditure, in history, forecasts and projections over a period of at least 60 years, is displayed in this graph as a percentage of GDP. While the NZSF model uses Net NZS as an input, aggregate gross NZS expenses have to be projected in order to generate the net values. Because it is the gross NZS data that the media, politicians, government officials, academics and the public are usually interested in, this graph is made available with every update of the NZSF model. All data in the graph comes from <i>NZS data</i>.</p>

5 Information about the *Input* worksheet

There are clear descriptions of what each input in this worksheet represents. Any cell shaded yellow either requires a numerical input or, in some cases, can be left with no entry. The model already contains the standard inputs for the latest EFU in these yellow-shaded cells.

Any of the standard inputs in yellow-shaded cells **can be altered** in order to model different future outcomes, other than those produced by the standard assumptions, for the NZSF. The standard assumptions can be restored by picking them up from *Standard inputs*.

Forecasts are made for an Economic and Fiscal Update (EFU), produced by the Treasury. These are usually published twice a year, at the Budget (BEFU) in May and the Half Year (HYEFU) in December, although there can be extra forecast rounds done, such as the Pre-election version (PREFU), or in response to a severe economic or fiscal change.

The normal EFU forecast period comprises the partially complete fiscal year in which the forecast is made, plus the next four fiscal years. Outturns, which are known results that have occurred, are recorded and forecasts are made, for variables that relate to the NZSF, by either the Treasury, MSD or the NZSF itself.

Projections are made for the fiscal years beyond the forecast period. With the important exceptions of the long-run input tracks of GDP and aggregate net NZS expenditure, which are produced by separate Treasury models and make significant use of the demographic projections of Stats NZ, the projections in the NZSF model are produced within the model itself. These projections are made in *Model*, and they are strongly influenced by both the forecasts from which they arise and the assumptions used in their calculation.

5.1 Inputs required for this version of the model to operate properly

There are two key inputs to the NZSF model that, were they changed or not provided, would at best give incorrect outputs from the model's logic or at worse mean that the model would not operate properly at all. For this reason they are listed in *Input*, but they are not shaded yellow to indicate that they are values that anyone using the model has the option to change. There is also an explanation next to the heading outlining why they should not be altered.

First projected year in the model is a marker value for many formulae in *Model* to signal when they should switch, from modelling logic appropriate to forecast years, to that designed for projected years. It should not be changed, because it is specific to the latest forecast horizon used in the model, and the model does not operate properly if it is altered.

Opening balance, in billions of dollars, of NZSF in first forecast year is the closing balance of the NZSF in the last fiscal year for which there are known outturns. This will be the last year displayed in *NZSF history*. As the closing balance of the NZSF in any year depends on the one in the previous year, this is needed to start off the annual calculations of the NZSF closing balance in *Model*. While a different figure can be put in and the model will still produce forecast and projected variables of the NZSF, these would not arise from the true base of the NZSF. For this reason it is not recommended to change this input.

5.2 Components of before-tax annual rate of return used in projected years

Beginning annual rate of return (before tax) for projections is the starting assumption, in projected years, for the annual rate of return on the assets of the NZSF before tax is removed. It is based on the parameters in the *Statement of Estimated Fund Performance* section of the annual [Statement of Performance Expectations](#) published by the NZSF. This should be viewed as a long-term, or equilibrium, expected gross (before tax) return rate for the NZSF, rather than as a forecast rate for any given year.

It is not realistic to try to predict the annual rises and falls in the return rate on NZSF assets as economic and financial market conditions fluctuate over the 40 year horizon applied in the legislated contribution rate formula. For this reason it is more practical to apply an average return rate over this period that smooths out the peaks and troughs that will undoubtedly unfold. This assumption is updated annually when the NZSF update their *Statement of Estimated Fund Performance*.

Despite this reasoning about applying an equilibrium return rate over the projected years, the next two input variables allow the initial assumed return rate for the NZSF to be gradually and linearly reduced over time. This acknowledges that, as the NZSF grows in size, it will become increasingly difficult for it to outperform the average market return rate.

Maximum reduction to the annual return rate over projections sets the maximum amount of percentage points by which the starting return rate assumption is lowered over projected years. No reduction can be applied in the modelling by setting this to zero.

Projected year in which reduction to the annual return rate begins sets the projected fiscal year in which the reduction to the initial assumed return rate for the NZSF commences.

The standard assumption used, for the beginning of the reduction to the gross annual rate of return initially applied in projections, is 20 years beyond the last completed fiscal year of NZSF outturns. This is based on the table with equilibrium expectations of components of the return rate for the NZSF over horizons of one year, ten years and 20 years, in the *Measuring our Performance* section of the latest [Statement of Intent](#) published by the NZSF.

Time period, in years, over which the reduction occurs in equal annual steps defines the period over which the reduction occurs. This means that this assumption, in combination with the maximum reduction assumed, also sets the annual rate at which the reduction is applied. Once the maximum reduction is attained, the assumed NZSF return rate stabilises, in all later projected years, at the assumed initial return rate minus the maximum reduction.

5.3 Other key assumptions used in projected years

Tax rate on earnings in projected years sets the annual tax rate assumed for the NZSF in projected years. The NZSF pays tax on its investment earnings, and so the gross returns in projected years are multiplied by this assumption to produce projections of tax payments.

This assumption is based on the annual tax rate applied by the NZSF analysts in producing their forecasts of tax on NZSF earnings. They take into account the New Zealand company tax rate, but also allow for lower tax rates on some of the overseas investments of the NZSF.

This assumption goes beyond just calculating projected NZSF tax payments. The return rate used in the legislated contribution rate formula is the **after tax return rate** on the assets of the NZSF, not the gross rate. As a consequence, this assumption plays an important role in calculating capital contributions to, and withdrawals from, the NZSF.

Funding horizon (number of years) specifies the length of the funding horizon over which the contribution rate formula is applied. The standard value of 40 years is the period specified in the legislated formula.

The longer the period specified the more the differential between what current taxpayers contribute to the NZSF and what future taxpayers receive from it ie, the greater the degree of tax smoothing that the NZSF produces between generations of taxpayers.

Because the modelling applies a matrix related to the number of years in the funding horizon, there are limits to the alternative values that can be chosen. The modelling logic will not be properly applied if anything other than a positive, integer value is assumed for the funding horizon. The minimum choice is one and the maximum is 80. A one year funding horizon quickly reverts to the current pay-as-you-go system of funding NZS. The maximum of 80 is due to requiring some limit on the number of rows in the matrix used in the *Model* worksheet.

5.4 Constraints to the logic of the model

There are several of these constraints that override the NZSF model logic. They are only applied to variables in the forecast years. The Treasury receives forecasts for the NZSF of earnings after non-tax expenses, tax on these earnings, and other movements in reserves from the NZSF analysts. These are for the current fiscal year and the next four at each EFU. From these forecasts the annual rate of return and the tax rate applied in each forecast year can be calculated and these are also included as forecast year constraints.

Because the capital contributions to the NZSF depend on the updated 40 year projections beyond each year, the NZSF analysts cannot know what these will be when they are making their forecasts for the same EFU round. However, they require some estimate of the capital contributions in each forecast year, in order to make forecasts of earnings for the same year, and so they normally apply the values from the previous edition of the NZSF model.

The Treasury replaces these capital contribution values from the NZSF, normally with the NZSF model's updated estimate for the forecast year. There are also times when decisions have been made by the government of the day to commit pre-determined levels of capital contributions in one or more forecast years. In these situations the planned contributions will be used in any forecast year for which they have been made.

Replacing the NZSF analysts' capital contribution forecast in any year would, at least in theory, change the forecasts of earnings and tax in that year. However, these are not adjusted from the values provided by the NZSF analysts. This is because, apart from the first, partially complete forecast year, these forecasts of earnings and tax are based on long-run assumptions of return rates and tax rates, and as such portray more of an average over forecast years and beyond, rather than the most likely outcome in any given year. Because of this, adjusting them for the usually moderate changes to the capital contributions in forecast years would only be introducing a degree of false precision to the forecasts.

The capital contribution will normally be constrained in the first forecast year, even if no constraints are applied in later years. The capital contribution for this first fiscal year is determined by the NZSF model in the Budget that precedes the start of this fiscal year. Once determined, it is communicated to the NZSF for their investment planning, and they receive it in monthly instalments throughout the fiscal year. For this reason it is not usually changed at future updates or in alternative scenarios.

One-off endowments are not a variable forecast by the NZSF analysts, but rather these entries allow for any one-off endowments, beyond the capital contribution, to be included in the NZSF modelling over forecast years. If no such endowments are desired in any of these years, which is the standard assumption, then these cells should be left blank.

Any value entered in the yellow-shaded cells in this section will be used by the NZSF model. It is only when a cell in this section is **left blank**, ie, no value is entered for the variable for that forecast year, **that the model will calculate the value for that year.**

Replacing one or more of the standard assumptions in the yellow-shaded cells in *Input* with alternative values, and filling in blank yellow-shaded cells with values, are the main means of producing **alternative scenarios** to the central NZSF forecast and projection for that EFU.

5.5 Nominal GDP and aggregate net NZS expenditure

The contribution rate formula in the *Model* worksheet requires long-term projections of GDP and Net NZS from the latest EFU forecast base.

Due to the truly long-run nature of these tracks, they are replicated in the *Standard inputs* worksheet. This ensures that they can be easily restored if they are altered to run scenarios.

6 Information about the *Model* worksheet

Model is the worksheet that calculates the *Contribution rate* in each year, produces the projected tracks of the NZSF and is the data source for most of the graphs displayed.

6.1 Year ending June

This variable serves both as a marker for some calculations in this worksheet and to list the fiscal year that each column represents. The first fiscal year in this series is Year 0 and is the last year of completed data before the forecast years begin. It is calculated by subtracting one from the first forecast year listed in the *Constraints to the logic of the model* section of *Input*. All subsequent years are simply calculated by adding 1 to the previous year's value.

6.2 Year index

The numbered years of the model are not used in any calculations. They are listed chiefly to show, for any modelled year, how many years it is beyond the last completed fiscal year of NZSF outturns, referred to in this guide as Year 0.

6.3 Inputs and assumptions

The **Nominal GDP** and **Net NZS expenditure** tracks are directly picked up from *Input*.

The formula for **Reduction of return rate per year** first checks:

- if the year involved is before **Projected year in which reduction to the annual return rate begins** in *Input*, and
- if it is, then it sets this variable to zero percent, but
- if it is not, then it calculates the sum of the previous year's value of this variable added to the maximum reduction divided by the time period in years over which the reduction occurs, and selects the smaller of this sum and the maximum reduction.

Both the maximum reduction and the time period come from *Input*. This modelling logic ensures that, from the projected year in which the reduction is first applied up until the projected year in which the maximum reduction is attained, the reduction will increase in equal annual increments. Beyond that the annual reduction remains at the maximum value.

Reduction of return rate per year = IF(*Year ending June* < *Projected year in which reduction to the annual return rate begins*, 0, MIN[Previous year's value of *Reduction of return rate per year* + *Maximum reduction to the annual return rate over projections* ÷ *Time period, in years, over which the reduction occurs in equal steps, Maximum reduction to the annual return rate over projections*])

For example, if the maximum reduction is 1% over a 40 year horizon, meaning the annual reduction rate is $1\% \div 40 = 0.025\%$, and this is first applied in the year 2034/35, then:

- this variable will be 0% up to and including 2033/34
- $0\% + 0.025\% = 0.025\%$ in 2034/35, $0.025\% + 0.025\% = 0.050\%$ in 2035/36, $0.050\% + 0.025\% = 0.075\%$ in 2036/37, and so on

- until reaching 1% in 2073/74, the 40th year of the interval beginning with 2034/35, and
- then remaining at 1% in all later projected years.

6.4 Forecast year variables

For the variables **Capital contribution in excess of net NZS expenditure**, **Earnings on assets**, **Tax on earnings**, **Other movements in reserves**, **Annual rate of return assumed in forecast**, **Annual tax rate assumed in forecast** and **One-off endowment** the formula checks, via the ISBLANK formula in Excel, whether or not a value has been entered for that year in the appropriate yellow-shaded cells in *Input*. If it has, then that value is used. Otherwise, the formula applies an alternative option depending on the variable involved.

Table 3 – Alternative options for calculating forecast year variables

Variable	Alternative option if no value has been entered in <i>Input</i>
Capital contribution in excess of net NZS expenditure	Use that fiscal year's calculated value of Required capital contribution as nominal \$ value from a row further down in <i>Model</i> , rounded to three decimal places.
Earnings on assets	Calculate as that year's values of the after-tax earnings divided by one minus the tax rate. The division is done in order to convert the earnings to a gross (before tax) return, as the earning rates used in the model are mainly after tax. The after-tax earnings are calculated as the previous year's Closing NZSF balance multiplied by that year's After-tax nominal annual rate of return <i>r</i>* on NZSF assets , added to that year's Capital contribution multiplied by the After-tax in-year compound return <i>w</i> with monthly rests . The result is rounded to three decimal places, and all of the variables used come from calculated variables in <i>Model</i> .
Tax on earnings	Calculate as that year's value of Earnings on assets multiplied by that year's value of Tax rate on annual earnings . The result is rounded to three decimal places, and the two variables come from calculated variables in <i>Model</i> .
Other movements in reserves and One-off endowment	Set to zero.
Annual rate of return assumed in forecast	Use Beginning annual rate return (before tax) for projections from <i>Input</i> .
Annual tax rate assumed in forecast	Use Tax rate on earnings in projected years from <i>Input</i> .

6.5 Calculated inputs

In the following table of descriptions of a number of input variables calculated in *Model*, unless specifically stated otherwise it should be taken that any variable used in a calculation relates to the same year for which the calculated variable is being produced.

The abbreviation COYT, standing for “check of year type”, should be taken to mean the formula uses an IF statement that checks whether the fiscal year involved is a forecast year or a projected year. It does this by calculating whether, for the year in which the calculation is made, the value of **Year ending June** is less than that of **First projected year in the model** in *Input*. If it is then the year involved is a forecast year, but otherwise it is a projected year.

Different types of return rates used in *Model* have distinguishing symbols in their names, such as r and r^* . This allows them to be differentiated from one another, and also makes it easier to reference the return rate in other variable names.

Table 4 – Description of calculation method of Calculated inputs variables

Calculated variable	Description of calculation method of variable
Before-tax nominal annual rate of return r on NZSF assets	A COYT is done by the formula. For a forecast year the value of Annual rate of return assumed in forecast is used. For a projected year the value of Beginning annual rate of return (before tax) for projections from <i>Input</i> minus Reduction of return rate per year is calculated.
Tax rate on annual earnings	A COYT is done by the formula. For a forecast year the value of Annual tax rate assumed in forecast is used. For a projected year the value of Tax rate on earnings in projected years is used from <i>Input</i> .
After-tax nominal annual rate of return r^* on NZSF assets	Calculated as Before-tax nominal annual rate of return r on NZSF assets multiplied by (1 minus Tax rate on annual earnings).
After-tax nominal annual rate of return on NZSF assets in form $(1+r^*)$	Calculated as one plus After-tax nominal annual rate of return r^* on NZSF assets . This form is useful for some of the matrix calculations used in this worksheet.
After-tax in-year compound return v with fortnightly rests	Calculated using the After-tax nominal annual rate of return r^* on NZSF assets variable. The derivation of the formula used is relatively complicated and so is explained in detail in Appendix Two to this guide.
Net NZS multiplied by after-tax in-year compound fortnightly return	Calculated as Net NZS expenditure multiplied by (One plus After-tax in-year compound return v with fortnightly rests). This form is useful for the calculation of the Contribution rate - Required contribution plus net NZS expenditure as % of nominal GDP variable.
After-tax in-year compound return w with monthly rests	Calculated using the After-tax nominal annual rate of return r^* on NZSF assets variable. As for the variable v , more detail about w is in Appendix Two to this guide.
Nominal GDP multiplied by after-tax in-year compound monthly return	Calculated as Nominal GDP multiplied by (One plus After-tax in-year compound return w with monthly rests). This form is useful for the calculation of the Contribution rate – Required contribution plus net NZS expenditure as % of nominal GDP variable.

6.6 Calculations related to the legislated contribution rate

The variable **Contribution rate – Required contribution plus net NZS expenditure as % of nominal GDP** is the fundamental calculation in the NZSF model. Everything else in the NZSF model either serves directly or indirectly as an input to the calculation of this variable, or is calculated from this variable, again either directly or indirectly.

However, the variable label in the NZSF model is rather long to keep referencing. Henceforth in this guide, including in the appendices, this variable will be referred to simply as the **Contribution rate**. It is also the variable denoted as a in [Section 43](#) of the 2001 Act.

The formula used to calculate **Contribution rate** is based on the requirements set out in Section 43, under the heading *Amount of required annual capital contribution*. It is a fairly complicated formula, involving summations and products over a 40 year horizon, and requiring matrix multiplications in the Excel formula that performs the necessary calculation.

For these reasons the formula is not described here. **Rather Appendix One to this guide explains it in detail.** It is recommended to read this appendix in order to better understand what the legislated formula in Section 43 means with respect to the logic of the NZSF, and how it translates to the mathematical formula used to calculate **Contribution rate**.

The derivation of the annualised rates of return for capital contributions or withdrawals and for Net NZS payments are explained in Appendix Two to this guide. These annualised rates are used, for each year involved in the **Contribution rate** calculation, in the first of the remaining years of the funding horizon.

The conversion of the **Contribution rate** formula to a calculation in *Model* and the matrix involved in this calculation is fully described in Appendix Three to this guide.

For a better comprehension of the **Contribution rate** formula, its components, and how it is calculated by the NZSF model, it is recommended to read all three appendices to this guide.

Table 5 – Description of calculation method of Contribution rate variables

Calculated variable	Description of calculation method of variable
Required-capital contribution plus net NZS expenditure in nominal dollars	Calculated as Nominal GDP multiplied by Contribution rate .
Required-capital contribution in nominal dollars	Calculated as Required capital contribution plus net NZS expenditure in nominal dollars minus Net NZS expenditure .
Required-capital contribution as % of nominal GDP	Calculated as Required capital contribution in nominal dollars divided by Nominal GDP .

The **Contribution rate** formula in the NZSF model involves two matrix multiplications:

- The first matrix multiplication is, for each year of the funding horizon, **Net NZS multiplied by after-tax in-year compound fortnightly return** multiplied by a vector of the remaining years of the funding horizon of **After-tax nominal annual rate of return on NZSF assets in form (1+r*)**.
- To the result of this matrix multiplication, which is a single number, is added another term. This is the previous year's value of **Closing NZSF balance**, which is calculated in a row further down in *Model*, multiplied by the product across all the years of the funding horizon of the values of **After-tax nominal annual rate of return on NZSF assets in form (1+r*)**. This summation forms the numerator of the formula.
- The second matrix multiplication, which produces the single number denominator of the formula, is, a repeat of the first one, with the exception that it uses **Nominal GDP multiplied by after-tax in-year compound monthly return in place of Net NZS multiplied by after-tax in-year compound fortnightly return**.

The **Contribution rate** formula in the NZSF model is the equivalent of that derived and shown at the end of Appendix One.

Using the matrix explained in Appendix Three, the formula produces the most important output of, and the fundamental variable in, the NZSF model, which is the **Contribution rate**.

6.7 Calculated NZSF variables in nominal dollars

Most of the variables in this section of the NZSF model are calculated by relatively simple formulae.

The abbreviation COYT, as applied in Table 4, is also used here.

Table 6 – Description of calculations of nominal dollar values of NZSF variables

Calculated variable	Description of calculation method of variable
Actual capital contribution plus net NZS expenditure	A COYT is done by the formula. For a forecast year the value of Capital contribution in excess of net NZS expenditure is added to Net NZS expenditure . For a projected year the value of Required capital contribution in nominal dollars is added to Net NZS expenditure .
less Net NZS expenditure	Set to Net NZS expenditure .
Capital contribution	Calculated as Actual capital contribution plus net NZS expenditure minus less Net NZS expenditure .
Gross earnings on NZSF assets	A COYT is done by the formula. For a forecast year it is set to Earnings on assets . For a projected year the value of Net earnings on NZSF assets , calculated a further two rows down, is divided by (One minus Tax rate on annual earnings).
less Tax paid on earnings on NZSF assets	A COYT is done by the formula. For a forecast year it is set to Tax on earnings . For a projected year the value of Gross earnings on NZSF assets is multiplied by Tax rate on annual earnings .
Net earnings on NZSF assets	A COYT is done by the formula. For a forecast year it is calculated as Gross earnings on NZSF assets minus less Tax paid on earnings on NZSF assets . For a projected year it is calculated as a sum of two terms. The first term is the previous year's value of Closing NZSF balance , calculated a further four rows down, multiplied by After-tax nominal annual rate of return r^* on NZSF assets . The second term in the sum is Capital contribution multiplied by After-tax in-year compound return w with monthly rests .
One-off endowment and Other movements in reserves	Set to One-off endowment and Other movements in reserves respectively in the <i>Forecast year variables</i> section of <i>Model</i> , meaning that both variables take on zero values in projected years.
Closing NZSF balance	Calculated as the summation of previous year's value of this same variable, Capital contribution , Net earnings on NZSF assets , One-off endowment and Other movements in reserves .

The reason that **Gross earnings on NZSF assets** is calculated, in projected years, from **Net earnings on NZSF assets** is because the earnings on the capital contribution in projected years are calculated using an after-tax return.

This is because no before-tax version of this return rate is calculated in the NZSF model. Consequently all earnings in projected years have to be calculated on an after tax basis, and then the total after-tax earnings converted back to gross, or before tax, earnings.

6.8 Variables related to the NZSF as percentages of nominal GDP

Every variable in this section is calculated by dividing the nominal dollar value of the variable with the same name in this worksheet by Nominal GDP and expressing this as a percentage.

6.9 Matrix used in the calculation of the key NZSF variable *Contribution rate*

The purpose of this matrix, the dependency of its number of utilised rows on the funding horizon being modelled, and the logic of the formula built into each of its cells is described in Appendix Three to this guide.

Appendix One: Explanation of the Contribution rate formula

The legal wording in [Section 43](#), concerning the *Contribution rate*, is quite lengthy, and so possibly a little difficult to follow. What it actually means in practice is as follows.

The *Contribution rate*, in the year that it is calculated for, is the combination, as a percentage of GDP, of the capital contribution (or, in later years, withdrawal) and Net NZS expenditure.

Of more relevance to the tax smoothing role is this interpretation of *Contribution rate*:
If this percentage of GDP was provided by taxpayers in each year of the funding horizon, with, in each year, any amount remaining after paying for Net NZS going to the NZSF and any deficit taken out of the NZSF, then this would cover the cost of Net NZS over this period. However, at the end of this funding horizon period, the funds of the NZSF would be depleted.

Tax smoothing function of the NZSF

A numerical example may help to make the above interpretation clearer.

Suppose in a particular year, call it Year 1, the *Contribution rate* is calculated to be 5.00% and in this year Net NZS is 4.60% of GDP. This means that in this year the NZSF will receive a capital contribution equal to $5.00\% - 4.60\% = 0.40\%$ of GDP.

Net NZS gradually rises over the next 40 years, reaching 5.00% in year 6 and 7.00% in Year 40, the final year of the 40 year funding horizon. If taxpayers provide 5% of GDP to the NZSF in each of these 40 years, then up until Year 6 the NZSF receives capital contributions, due to Net NZS being under 5% of GDP. From Year 7 to year 40, Net NZS is above 5% of GDP. In these years withdrawals are taken out of the NZSF in order to make up the difference between whatever Net NZS is equal to as percentage of GDP in the year and 5% of GDP.

Taxpayers across the full 40 year horizon contribute 5% of GDP in each year to pay for Net NZS, meaning taxpayers in the early years pay more than the cost of net NZS, while those in the later years do not fully fund Net NZS. The tax smoothing function of the NZSF relates to how this process allows current taxpayers to assist future generations of taxpayers, because an ageing population structure will drive up the cost of NZS as a percentage of GDP.

The reason that the value of the *Contribution rate* keeps rising

It is not obvious why the value of the *Contribution rate* keeps rising over the years in the NZSF model, given that the legislated formula is apparently supposed to produce a flat percentage of GDP level of funding of Net NZS.

There are a few reasons for this. Foremost among these is that, if this percentage of GDP were retained as the *Contribution rate* value in every year, then the funds of the NZSF would be depleted at the end of the 40 year horizon. The NZSF is not designed to be depleted in 40 years' time, but rather is intended to act as a tax smoothing vehicle for considerably longer than that. Consequently the contribution rate is recalculated each year, and because Net NZS is projected to continue to rise as a percentage of GDP over the next century (and likely beyond that too), the recalculated value of the *Contribution rate* also keeps rising.

This does not prevent the NZSF performing a tax smoothing role, because the *Contribution rate* does not rise at the same rate as Net NZS as a percentage of GDP does. Hence the lines still cross at a future point, as shown in the *Contribution rate* graph in the NZSF model.

The mathematical version of the legislated *Contribution rate* formula

The mathematical interpretation of the *Contribution rate* formula is derived by solving the following equation:

$$E_0 \left[B_{j-1} \prod_{t=1}^H (1 + r_{j+t-1}) + \sum_{t=1}^H k_j G_{j+t-1} (1 + m_{j+t-1}) \prod_{i=t+1}^H (1 + r_{j+i-1}) - \sum_{t=1}^H P_{j+t-1} (1 + f_{j+t-1}) \prod_{i=t+1}^H (1 + r_{j+i-1}) \right] = 0$$

where:

$E_0[\dots]$ = expected value given information available at the beginning of year j

B_{j-1} = closing balance of the NZSF in year j-1, which equals opening balance in year j

P_t = aggregate net (of tax) New Zealand Superannuation (NZS) expenses in year t

G_t = nominal GDP for year t

H = time horizon for the calculation (while it can be varied, it has always been 40 years)

k_j = contribution rate as a proportion of nominal gross domestic product (GDP) in year j

r_t = annual after-tax rate of return on the NZSF in year t

m_t = annualised after-tax return on the NZSF in year t to reflect the monthly pattern in which capital contributions are received by the NZSF during the year

f_t = annualised after-tax return on the NZSF in year t to reflect the fact that NZS entitlements are paid to superannuitant recipients in fortnightly instalments.

In deriving the solution for the contribution rate, k_j , in a given year j the following logic will drop the expected value notation, $E[\dots]$. However, it should always be appreciated that these are forward looking calculations. They are based on forecasts and projections of the NZSF balance, GDP and aggregate net NZS expenditure, rather than known, actual values.

Rearranging the smoothing equation that equates to zero gives:

$$\sum_{t=1}^H k_j G_{j+t-1} (1 + m_{j+t-1}) \prod_{i=t+1}^H (1 + r_{j+i-1}) = \sum_{t=1}^H P_{j+t-1} (1 + f_{j+t-1}) \prod_{i=t+1}^H (1 + r_{j+i-1}) - B_{j-1} \prod_{t=1}^H (1 + r_{j+t-1})$$

As k_j is not changing as t changes in this formula it can be taken outside the term on the left hand side of this equation and then the right hand side of the equation can be divided by the remaining multiplier term to give:

$$k_j = \frac{\sum_{t=1}^H P_{j+t-1} (1 + f_{j+t-1}) \prod_{i=t+1}^H (1 + r_{j+i-1}) - B_{j-1} \prod_{t=1}^H (1 + r_{j+t-1})}{\sum_{t=1}^H G_{j+t-1} (1 + m_{j+t-1}) \prod_{i=t+1}^H (1 + r_{j+i-1})}$$

This is the formula that calculates the *Contribution rate* in any year. It lies at the heart of all calculations in the NZSF model.

Explanations of the terms in the *Contribution rate* formula

The *Contribution rate* formula involves the:

- two key input tracks into the model, that of forecast and projected fiscal year values of nominal GDP and aggregate net NZS expenditure
- projected value of the NZSF at the time that the calculation is done, and
- after-tax return rates on all of these components in each year of the funding horizon that stretches out from the calculation year.

$$\sum_{t=1}^H P_{j+t-1} (1 + f_{j+t-1}) \prod_{i=t+1}^H (1 + r_{j+i-1})$$

This term is the summation, over all of the years in the funding horizon, of the annual aggregate net NZS expense amounts, with each multiplied by the after-tax return rates on it over the remaining years of the funding horizon. Depending on when each payment occurs in the funding horizon, it has a certain number of years remaining in which to generate returns.

In the year in which each net NZS payment occurs, because superannuitants receive NZS on a fortnightly basis, the in-year return generated is calculated via an annualised return rate that reflects this fortnightly timing. Appendix Two to this guide has more detail about this. It suffices here to state that the first fortnightly payment of the year has the entire rest of that year to generate a return, one made halfway through the year only has the remaining half of the year to do this, and the last fortnightly payment of the year has no time left in that year. The term $P_{j+t-1}(1+f_{j+t-1})$ represents the value of the aggregate net NZS expenditure and the in-year return generated on it for each year t in the funding horizon.

For the remaining years of the funding horizon, for each net NZS payment, the full annual after-tax return rates apply. If that payment occurs in the calculation year, and a 40 year funding horizon is used, then there will be 39 years left for the payment and its in-year return to generate returns. If the payment occurs in the 25th year of this 40-year funding horizon, there will only be 15 years left to generate returns on it and its in-year return. If it occurs in the 40th year, there will be no years left in the funding horizon to generate returns on the payment and its in-year return.

The product expression $\prod_{i=t+1}^H (1 + r_{j+i-1})$ represents the cumulative return generated on each annual net NZS payment and its in-year return over the funding horizon. This multiplier starts in year $t+1$ for a net NZS payment made in year t of a funding horizon of H years, and ends in year H . This means that no multiplier is calculated for year H , as it would start in the year after which it was meant to finish. In effect the value of the multiplier is one in year H , the final year of the funding horizon.

When this is all accumulated via the summation term shown, the result is a single value. This value represents the sum of all of the annual aggregate net NZS expenditures, as well as the returns generated upon each one, over the funding horizon beginning in the year for which the *Contribution rate* is calculated.

$$B_{j-1} \prod_{t=1}^H (1 + r_{j+t-1})$$

This is the other term in the numerator of the formula. B_{j-1} represents the NZSF closing balance in the year previous to the one for which the *Contribution rate* is calculated ie, the value of the NZSF at the beginning of the *Contribution rate* calculation period. This has the entire funding horizon in which to generate returns, so the product expression covers from year $t = 1$ to $t = H$. The product expression has been described above and, when applied as a multiplier to the NZSF closing balance term, allows for the combination of this closing balance and the returns generated on it over the funding horizon to be calculated.

This expression is subtracted in the numerator from the one related to the net NZS payments and their returns. This is because, with respect to paying for net NZS expenditure over the funding horizon, which the *Contribution rate* formula is based upon, the NZS expenses and their returns build up this cost, while the size of the NZSF and its returns reduce it.

$$\sum_{t=1}^H G_{j+t-1} (1 + m_{j+t-1}) \prod_{i=t+1}^H (1 + r_{j+i-1})$$

This is the expression in the denominator of the formula. It is very similar to the one described for the aggregate net NZS expenditure terms over the funding horizon, except for two key differences. The first of these is that the net NZS values in each year, P_{j+t-1} , are replaced by nominal GDP values in these same years, G_{j+t-1} . The second difference is that the annualised return rates that reflect the fortnightly timing of the NZS payments, f_{j+t-1} , are replaced by annualised return rates, m_{j+t-1} , that reflect the monthly timing of the capital contributions to, and in later years withdrawals from, the NZSF.

The logic behind this expression is no different to that explained for the similar one that relates to the NZS payments, except that the in-year returns are calculated on a monthly basis rather than a fortnightly one.

It may seem odd to think of GDP amounts generating returns, but that is not what this expression represents. This term is in the denominator because it was divided out from being a multiplier of the *Contribution rate* k_j , in order to isolate k_j as the subject of the formula. When multiplied by the constant value k_j , the annual GDP amounts in this expression give the total amounts contributed by taxpayers to fund NZS in each year of the funding horizon. It is these funding amounts that are generating returns within the year in which they are paid and in all remaining years of the funding horizon, not GDP itself.

Appendix Two: Derivation of the annualised return rates formulae

The capital contributions to the NZSF (and, presumably, the withdrawals once they begin) and the Net NZS payments are made in instalment periods that are more frequent than once a year. The contributions are made monthly, while superannuitants receive their NZS payment every fortnight.

Annualised return rates are only applicable, for either an aggregate net NZS payment or a capital contribution or withdrawal, to the year in the funding horizon in which this payment, contribution or withdrawal occurs. This is because in all later years of the funding horizon these annual amounts have the entire year in which to generate returns.

For this reason also, annualised return rates do not apply to the closing NZSF balance from the previous year to that for which the *Contribution rate* is being calculated. This amount has every year of the funding horizon in which to generate returns.

Derivation of the general annualised return rate formula

Imagine an amount P that is paid out in N equal instalments over the year. If the payment schedule was fortnightly, as with Net NZS payments, N would be 26^3 and if it was monthly, as is now the case for capital contributions, N would be 12.

If the annual rate of return in the first year is denoted by r , then the return rate s that compounds over N periods in the year to give the same return must satisfy:

$$(1+s)^N = 1 + r \quad \rightarrow \quad 1 + s = (1+r)^{1/N}$$

If each payment is made at the end of the payment frequency period, ie, the end of each fortnight or the end of each month, then the first payment has $N-1$ periods to generate a return with a return rate s in a compounding manner, the second payment has $N-2$ periods to do this, and so on until the $(N-1)$ th payment has 1 period left in the year to generate a return and the N th payment has no time left to do this.

Furthermore, each payment will be of amount $P \div N$, so that at the end of the year the value of the series of payments will be:

$$\sum_{j=0}^{N-1} \frac{P}{N} (1 + s)^j = P \sum_{j=0}^{N-1} \frac{1}{N} (1 + s)^j$$

³ About once in every dozen years there will be 27 fortnightly payments in the year, if the payment schedule adhered to is strictly every fortnight. While that is accepted as a reality, it is not accounted for in this formula, and is likely to be a very small source of inaccuracy. Likewise, if monthly payments occur at the end of each month, then there is a small inaccuracy caused by the fact that this does not result in twelve equally spaced payments over the year.

If z is the annual rate of return that would have produced the same value, at the end of the year, had the amount P been invested for the whole year, then:

$$P(1+z) = P \sum_{j=0}^{N-1} \frac{1}{N} (1+s)^j \rightarrow 1+z = \sum_{j=0}^{N-1} \frac{1}{N} (1+s)^j \rightarrow 1+z = \frac{1}{N} \sum_{j=0}^{N-1} (1+s)^j$$

However, from above, $1+s = (1+r)^{1/N}$ $(1+s)^j = [(1+r)^{1/N}]^j$ $(1+s)^j = (1+r)^{j/N}$

Therefore $1+z = \frac{1}{N} \sum_{j=0}^{N-1} (1+r)^{j/N}$

The summation term here is:

$$\frac{1}{N} + \frac{1}{N}(1+r)^{\frac{1}{N}} + \frac{1}{N}(1+r)^{\frac{2}{N}} + \frac{1}{N}(1+r)^{\frac{3}{N}} + \dots + \frac{1}{N}(1+r)^{\frac{N-2}{N}} + \frac{1}{N}(1+r)^{\frac{N-1}{N}}$$

This can further be expressed in the form:

$$\frac{1}{N} + \frac{1}{N}(1+r)^{\frac{1}{N}} + \frac{1}{N}[(1+r)^{\frac{1}{N}}]^2 + \frac{1}{N}[(1+r)^{\frac{1}{N}}]^3 + \dots + \frac{1}{N}[(1+r)^{\frac{1}{N}}]^{N-2} + \frac{1}{N}[(1+r)^{\frac{1}{N}}]^{N-1}$$

This is a geometric series of N terms, with constant multiplier $1/N$ and power term $(1+r)^{1/N}$ and so has a solution given by:

$$\frac{\frac{1}{N} \left[1 - \left\{ (1+r)^{\frac{1}{N}} \right\}^N \right]}{1 - (1+r)^{\frac{1}{N}}} = \frac{\frac{1}{N} [1 - (1+r)]}{1 - (1+r)^{\frac{1}{N}}} = \frac{\frac{1}{N} (-r)}{1 - (1+r)^{\frac{1}{N}}} = \frac{\frac{1}{N} (r)}{(1+r)^{\frac{1}{N}} - 1} = \frac{r}{N \left[(1+r)^{\frac{1}{N}} - 1 \right]}$$

Consequently $z = \frac{r}{N \left[(1+r)^{\frac{1}{N}} - 1 \right]} - 1$

In the simplified equation for the *Contribution rate* in year j , k_j , when the after-tax annual rate of return r is assumed to have stabilised, the annualised return rate for the fortnightly aggregate net NZS expenditure payments was designated as v while that of the monthly capital contributions (or withdrawals) was w .

The above derivation now shows that, for the case when the general annualised return rate z , for which the formula was derived:

- is the fortnightly annualised return rate v , which means that the number of compounding periods N is 26, then $v = \frac{r}{26 \left[(1+r)^{\frac{1}{26}} - 1 \right]} - 1$
- and when it is the monthly annualised return rate w , which means that the number of compounding periods N is 12, then $w = \frac{r}{12 \left[(1+r)^{\frac{1}{12}} - 1 \right]} - 1$

These are the two *After-tax in-year compound return* terms that are used in the NZSF model.

Appendix Three: Explanation of the matrix used in *Model*

Before describing in more detail the formula used in *Model* to calculate the *Contribution rate*, the large matrix at the bottom of the worksheet that is used in this formula is explained.

This matrix is used to produce the product terms $\prod_{i=t+1}^H (1 + r_{j+i-1})$ in the calculation of the *Contribution rate* in each year. The formula used in every cell of the matrix is identical. However, the output in each cell is dependent upon the calculations produced by the formula from the inputs that it receives for the particular row and column.

The matrix needs to have as many rows as there are years in the funding horizon chosen in *Input*. While there is no theoretical reason why this needs to be limited, for practical reasons only 80 rows are set up with the necessary formula in the NZSF model. This is why the maximum selection of the variable *Funding horizon (number of years)* in *Input* is limited to 80. At least one year is needed to calculate *Contribution rate*, so the minimum choice is one.

In column B of *Model* is a number index from 1, in line with the first year of calculations in the matrix, to 80. The formula in each matrix cell checks whether this index number, for the row that the cell is in, is less than, equal to, or greater than the funding horizon chosen in *Input*.

- If the index number is less than the number of years in the funding horizon, the value of the cell is calculated as the product of the variable *After-tax nominal annual rate of return on NZSF assets in form (1+r*)* over the remaining years of the funding horizon.
- If the index number is equal to the number of years in the funding horizon, the value of the cell is set to one.
- If the index number is greater than the number of years in the funding horizon, the value of the cell is set to zero. In this situation the worksheet cell is not utilised in the matrix, so is given a zero value simply to indicate that a formula is present in the cell.

In the final year of the funding horizon, it is the only year remaining in which to accumulate returns on the GDP and Net NZS terms in the *Contribution rate* formula. These are in-year returns calculated by the appropriate annualised return rates. The product terms relate to returns generated in years beyond the one in which the GDP or Net NZS values occur, and there are none of these years once the final year of the funding horizon is attained.

Consequently, when the index number is equal to the number of years in the funding horizon the matrix cell value is set to one. This effectively removes the product multiplier term from the two matrix multiplications, reducing:

- $\sum_{t=1}^H P_{j+t-1} (1 + f_{j+t-1}) \prod_{i=t+1}^H (1 + r_{j+i-1})$ to simply $\sum_{t=1}^H P_{j+t-1} (1 + f_{j+t-1})$, and
- $\sum_{t=1}^H G_{j+t-1} (1 + m_{j+t-1}) \prod_{i=t+1}^H (1 + r_{j+i-1})$ to simply $\sum_{t=1}^H G_{j+t-1} (1 + m_{j+t-1})$.

Matrix cell formula when number of years is less than the funding horizon

A numerical example may help to make the logic applied in this situation clearer. In this description the variable *After-tax nominal annual rate of return on NZSF assets in form* $(1+r^*)$ will be simply referred to as $(1+r^*)$.

Suppose the calculation of the *Contribution rate* is being done for the eighth year in the model following Year 0, that the funding horizon is the standard one of 40 years, and that the focus of this example is the 15th row of the matrix in this eighth year of the model.

The formula in the cell does a product calculation of the variable $(1+r^*)$ over all of the remaining years of the 40 year funding horizon. The formula uses the Excel function OFFSET, and there are five inputs to this function.

The first of these sets a cell in the worksheet that the next two inputs will be displaced from vertically and horizontally. This first input is always the worksheet cell containing the variable $(1+r^*)$ for the year in which the *Contribution rate* is being calculated. In this example that will be the eighth year so this starting point will be the cell in *Model* corresponding to the row containing the $(1+r^*)$ variable and the column that is labelled with the number 8 in the *Year index* row at the top of the worksheet.

The next input to OFFSET is a zero, because the formula is designed not to move up or down any rows from this starting point.

The third input is the number of the row index in the matrix, in column B beside the matrix. This moves the beginning of the matrix (or array as it is referred to in Excel terminology), defined by the last two inputs in the OFFSET function, by this number of columns from the starting point. In this example, this moves the beginning of the matrix to the column with the *Year index* 23 in the row containing the $(1+r^*)$ variable, because $8 + 15 = 23$.

The next input is an entry of one, meaning the number of rows to be covered in the matrix is just one, which effectively means it is a horizontal vector that is being defined.

The final input is the value of the number of years in the funding horizon minus the row index number in the matrix. In this example that has a value of $40 - 15 = 25$.

Hence, what is defined by the OFFSET function for this matrix cell in the eighth column and 15th row is a vector of the values of the $(1+r^*)$ variable, starting in the 23rd year following Year 0 and extending for 25 columns, which takes it up to and including the entry in the column with the *Year index* 47. (Remember that the 23rd year is the first year, so a 25 year interval ends 24 years after the 23rd year, not 25 years after it).

The PRODUCT function multiplies the 25 values in this vector together to give a single number. This single number is **a compound return rate**.

In this particular example, what is calculated in the eighth column and 15th row of the matrix is the compound return rate that is the result of the term $\prod_{i=t+1}^H (1 + r_{j+i-1})$ when $j = 8$ and $t = 15$. This is the product, where subscripts denote year numbers beyond Year 0, $(1+r_{23}^*)(1+r_{24}^*)(1+r_{25}^*)(1+r_{26}^*)\dots\dots(1+r_{44}^*)(1+r_{45}^*)(1+r_{46}^*)(1+r_{47}^*)$

What this product represents for the *Contribution rate* calculation in *Model*

In the particular example given, the compound return rate calculated in the matrix cell is applied to any amount that is generated in the 15th year of the 40 year funding horizon that begins in the eighth year after Year 0. This is the 22nd year after Year 0, and any amount generated in this year, be it GDP or Net NZS, has 25 years remaining in which to generate returns, before the funding horizon that began in the eighth year ends in the 47th year.

A **compound return rate** is what each cell in the matrix calculates **for its given row and column position**.

The column number equates to the fiscal year, as denoted by its *Year index* number in *Model*, for which the *Contribution rate* is calculated.

The row number, from column B of the matrix at the bottom of *Model*, equates to the number of years into the funding horizon that the year occurs to which the calculated compound return rate applies.

This means that the year for which the calculated compound return rate applies has *Year index* equal to the column number plus row number minus one.

The interval over which the compound return rate is calculated begins in the year following the one to which it applies, meaning that the interval starts in the year with *Year index* equal to the sum of the row and column numbers.

The number of years over which the compound return rate is calculated equals the funding horizon length minus the row number. As it applies to the year with *Year index* equal to the column number plus row number minus one, this means it must end in the year with *Year index* equal to the funding horizon length plus the column number minus one.

Some numerical examples may help to clarify the description above, where it will be assumed that the standard funding horizon of 40 years is being applied. However, there is nothing to stop a different choice of funding horizon being applied, and the matrix is set up to work for any whole number choice of funding horizon from 1 year through to 80 years.

The compound return rate calculated in the matrix cell in row 1, column 1:

- applies to the calculation of the *Contribution rate* in the year with *Year index* 1
- applies to the year with *Year index* = $1+1-1 = 1$
- is calculated over an interval that begins in the year with *Year index* = $1+1 = 2$
- is calculated for an interval of length $40-1 = 39$ years, and
- is calculated over an interval that ends in the year with *Year index* = $40+1-1 = 40$.

The compound return rate calculated in the matrix cell in row 23, column 14:

- applies to the calculation of the *Contribution rate* in the year with *Year index* 14
- applies to the year with *Year index* = 14+23-1 =36
- is calculated over an interval that begins in the year with *Year index* = 14+23 =37
- is calculated for an interval of length 40–23 =17 years, and
- is calculated over an interval that ends in the year with *Year index* = 40+14-1 =53.

The compound return rate calculated in the matrix cell in row 18, column 45:

- applies to the calculation of the *Contribution rate* in the year with *Year index* 45
- applies to the year with *Year index* = 45+18-1 =62
- is calculated over an interval that begins in the year with *Year index* = 45+18 =63
- is calculated for an interval of length 40–18 =22 years, and
- is calculated over an interval that ends in the year with *Year index* = 40+45-1 =84.

The main formula used in *Model* to calculate the *Contribution rate*

This uses the MMULT function of Excel, which allows matrix multiplication of arrays in Excel.

In this formula both matrix multiplications involve a row vector multiplied by a column vector, so result in a 1x1 matrix product, which of course means just a single number.

$$k_j = \frac{\sum_{t=1}^H P_{j+t-1} (1 + f_{j+t-1}) \prod_{i=t+1}^H (1 + r_{j+i-1}) - B_{j-1} \prod_{t=1}^H (1 + r_{j+t-1})}{\sum_{t=1}^H G_{j+t-1} (1 + m_{j+t-1}) \prod_{i=t+1}^H (1 + r_{j+i-1})}$$

First component
Second component

Third component

The first component of the *Contribution rate* formula calculates, via a matrix multiplication, the expression $\sum_{t=1}^H P_{j+t-1} (1 + f_{j+t-1}) \prod_{i=t+1}^H (1 + r_{j+i-1})$.

The row vector is expressed in Excel via an OFFSET function. Its starting point is the position of the variable *Net NZS multiplied by after-tax in-year compound fortnightly return* in the year that the *Contribution rate* is calculated for, and it extends out the number of columns equal to the length of the funding horizon.

For example, if the standard funding horizon of 40 years is used, this will be a 1x40 vector of the variable's values. It represents the $\sum_{t=1}^H P_{j+t-1}(1 + f_{j+t-1})$ section of the formula

The column vector is also expressed via an OFFSET function, applied to the large matrix of compound return rates from its first row in the same column from which the row vector begins. It has the same number of rows as the length of the funding horizon, and contains the compound return rates represented by the $\prod_{i=t+1}^H (1 + r_{j+i-1})$ section of the formula.

Each column entry in the row vector is multiplied by the corresponding row entry in the column vector via the matrix multiplication, eg, the 5th column entry of the row vector is multiplied by the 5th row entry of the column vector.

This matrix operation also adds up the sum of all of these multiplications to give a single number. This is the value of the entire expression $\sum_{t=1}^H P_{j+t-1}(1 + f_{j+t-1}) \prod_{i=t+1}^H (1 + r_{j+i-1})$.

The second component of the *Contribution rate* formula just requires the *Closing NZSF balance*, from the previous year to that for which the *Contribution rate* is being calculated, to be multiplied by the product of the $(1+r^*)$.variable over the length of the funding horizon.

This is the $B_{j-1} \prod_{t=1}^H (1 + r_{j+t-1})$ part of the *Contribution rate* formula with the product section of it calculated by the combination of PRODUCT and OFFSET functions. It is subtracted from the first component to calculate the numerator of the *Contribution rate* formula.

The final component, the denominator of the *Contribution rate* formula, is calculated in exactly the same way as the first component, except that it uses *Nominal GDP multiplied by after-tax in-year compound monthly return* instead of *Net NZS multiplied by after-tax in-year compound fortnightly return*.